15.5 Lecture: Triple integrals in rectangular coordinates

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3D Regions

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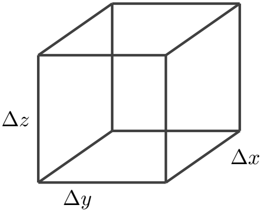

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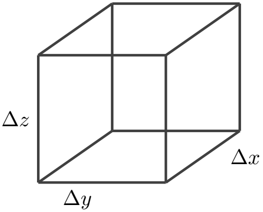
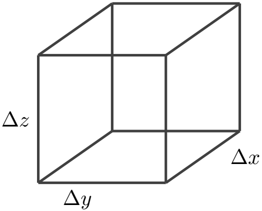


Then the integral of $f(x, y, z)$ over D is defined as in previous sections for double integrals as the limit of a Riemann sum and is denoted

$$\int\int\int_D f(x, y, z)dV$$

dV



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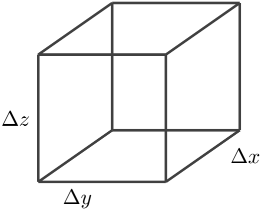

Since we're chopping up the region D into small rectangular solids, the volume differential will be the product of the differentials in each coordinate direction.
Usually, we'll use the orders

$$dV = dzdydx \quad \text{or} \quad dV = dzdxdy.$$

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Process for finding limits in *dzdydx* order (pg. 907-909)

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$$\int\int\int_D f(x,y,z)dV = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Volume

Definition

The volume of a closed, bounded region D is

$$\int\int\int_D dV.$$

As before with 2D regions and double integrals, we can calculate the volume of a 3D region D with a triple integral by putting $f(x, y, z) = 1$ inside the triple integral over D.

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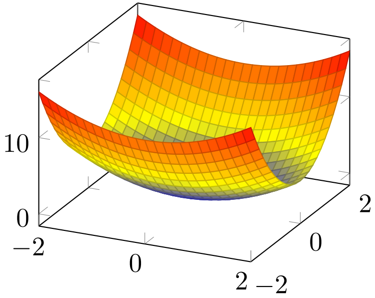

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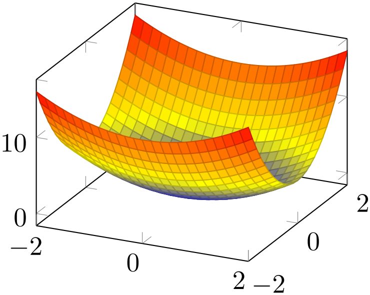
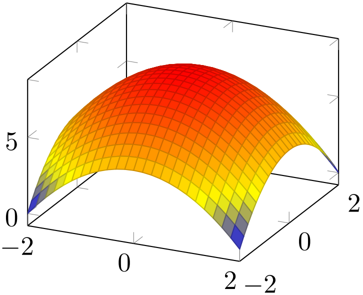


$g(x, y) = 8 - x^2 - y^2$



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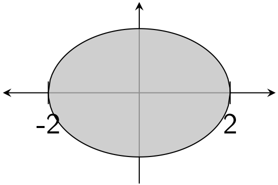

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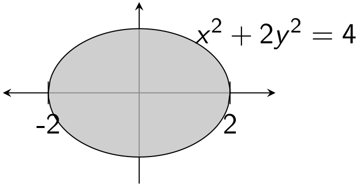
Thus $x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2$.

Finding the y-bounds

Now we look at the shadow of the region in the xy-plane.



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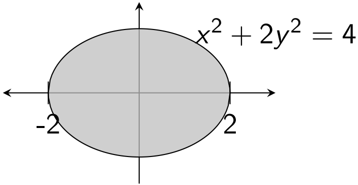

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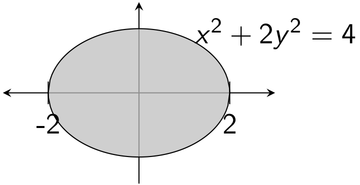

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Answer

$$-\sqrt{2 - \frac{x^2}{2}} \leq y \leq \sqrt{2 - \frac{x^2}{2}}$$

Finding the x-bounds

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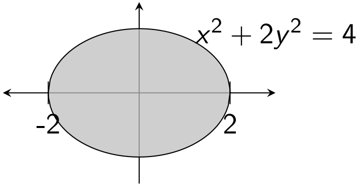

Question

What is the smallest or largest x can be?

Answer

$-2 \leq x \leq 2$

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Thus the volume is given by

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{2-\frac{x^2}{2}}}^{y=\sqrt{2-\frac{x^2}{2}}} \int_{z=x^2+3y^2}^{z=8-x^2-y^2} 1 \; dz \; dy \; dx$$